The propagation of small disturbances in hydromagnetics

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This paper is concerned with the propagation of small initial disturbances in a conducting gas under the influence of a uniform external magnetic field.

For a perfect conductor, there are three types of plane waves, each of which depends strongly on the angle at which the magnetic field is crossed. The modifying effects of finite conductivity are determined and, in the case of these waves, this is done uniformly for all angles. A general disturbance may be resolved into two parts, one of which satisfies a fourth-order equation and the other a fifth; for a perfect conductor these reduce to second- and fourth-order equations, respectively.

The free oscillations of the gas are examined when it is contained in a rectangular box, and, in particular, when the external field is very weak or very strong. For vanishingly weak fields the idealization of infinite conductivity proves to be inadequate. Finally, the initial-value problem is discussed.

1. Introduction

In this paper we consider the small perturbations of an electrically conducting inviscid gas at rest in the presence of a uniform external magnetic field. The main interest lies in fluids of high (but not infinite) conductivity, and in order to obtain all effects we do not neglect displacement currents.

The system of eleven first-order linear differential equations governing the motion possesses two integrals (invariants in time) so that the effective order is nine. First we discuss plane waves which, in the context of forced oscillations, have been previously considered by van de Hulst (1951) and by Baños (1955), but in each case with certain important limitations. The equations are found to divide into two sets, one leading to a quartic for the frequency as a function of the wavelength and the other to a quintic. Two of the roots of the quartic and one of the quintic give pure decay of which the main effect is the reduction of the initial electrical field, as measured in the local frame, to zero; this being required in a perfect conductor. For such a conductor the remaining two and four roots give the three possible modes of propagation of sinusoidal disturbances. One of these can be identified as an Alfvén wave; the other two cannot in general be distinguished except on the basis of velocity.† The damping effect of finite conductivity on these waves is determined (uniformly) for all inclinations of the wave front to the undisturbed magnetic field. For small inclinations the damping predominates in two of the modes and the disturbances decay exponentially.

Returning to the general perturbation equations, we show that, by suitable

† These correspond to Friedrichs's (1957) transverse, fast and slow disturbance waves.

transformation, these also can be split into sets of four and five. Each set leads, on elimination of all but any one of the independent variables, to a partial differential equation of the same order. These orders reduce to two and four, respectively, for a perfect conductor; the second-order equation then being a one-dimensional wave equation and the fourth-order equation exhibiting similar strong anisotropy in the direction of the external field.

We next examine a simple boundary-value problem in detail, namely, the standing waves in the fluid when it is confined in a rectangular box made of perfectly conducting material into which a uniform magnetic field perpendicular to two of the faces has been frozen. These waves, which consist of combinations of plane waves, are of three types: those corresponding to an infinity of frequencies associated with the Alfvén velocity, and two others distinguished by their symmetry properties about the median plane and having separate frequency equations. The resulting pair of distributions of frequencies are sketched for the two extreme cases of very weak and very strong external fields. In the former case the ordinary acoustic frequencies are obtained in the limit, half from one equation and half from the other, while all other frequencies tend to zero.

The limiting forms of the waves themselves for the two extreme cases are also discussed, and here we encounter two interesting phenomena for a vanishingly weak external field. First, the last two types of waves become indeterminate if the conductivity has been taken infinite, but tend to definite limits for any finite conductivity (which may then tend to infinity). Secondly, for all waves the tangential component of velocity is zero on the two walls perpendicular to the external field, while no such condition need be satisfied in the absence of an external field. Such initial discontinuities in the latter case cling to the walls as vortex sheets, while in the presence of a (non-parallel) external field they immediately move away. We conclude that, however large the conductivity of the fluid may be, the idealization of its being perfectly conducting is not good in the case of sufficiently weak external fields; and that however weak the field, a vortex sheet can only remain adjacent to walls which are parallel to the field.

We conclude with a discussion of the general initial-value problem and the determination of Fourier coefficients appropriate to the orthogonal system defined by the standing waves.

2. The equations of motion

We consider the motion of an electrically conducting inviscid gas in the absence of body forces and heat conduction. Further we shall assume that the material coefficients μ (permeability), ϵ (dielectric constant), and σ (conductivity) are constant. Then the equations governing the motion are

$$\left.\begin{array}{c} \mu\dfrac{\partial \mathbf{H}}{\partial t} = -\operatorname{curl}\mathbf{E}, \quad \dfrac{1}{\sigma}(\mathbf{J}-\rho_e\mathbf{v}) = \mathbf{E}+\mu\mathbf{v}\times\mathbf{H}, \\ \rho\dfrac{d\mathbf{v}}{dt} = -\operatorname{grad}p+\rho_e\mathbf{E}+\mu\mathbf{J}\times\mathbf{H}, \\ \dfrac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v} = 0, \quad \rho T\dfrac{dS}{dt} = \dfrac{1}{\sigma}(\mathbf{J}-\rho_e\mathbf{v})^2. \end{array}\right\} \qquad (1)$$

Here the temperature T and the specific entropy S are given functions of the pressure p and density ρ, while ρ_e (charge density) and $\mathbf{J}$ (current density) are related to the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ by

$$\rho_e = \epsilon \operatorname{div} \mathbf{E}, \quad \mathbf{J} = \operatorname{curl} \mathbf{H} - \epsilon \frac{\partial \mathbf{E}}{\partial t}.$$

The system (1) consists of eleven first-order equations for $\mathbf{E}, \mathbf{H}, \mathbf{v}, p, \rho$ and we only admit solutions for which

$$\operatorname{div} \mathbf{H} = 0, \tag{2}$$

this condition being satisfied at all times if it is satisfied at any one instant since $\partial \mathbf{H}/\partial t$ is solenoidal.

If the deviation from a given uniform state $\mathbf{E} = 0$, $\mathbf{H} = \mathbf{H}_0$, $\mathbf{v} = 0$, $p = p_0$, $\rho = \rho_0$ is small, so that we may neglect squares and products of disturbance quantities, the system (1) reduces to

$$\left.\begin{aligned} &(a) \quad \mu \frac{\partial \mathbf{H}}{\partial t} = -\operatorname{curl} \mathbf{E}, \quad (b) \quad \frac{1}{\sigma} \mathbf{J} = \mathbf{E} + \mu \mathbf{v} \times \mathbf{H}_0, \\ &(c) \quad \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad} p + \mu \mathbf{J} \times \mathbf{H}_0, \\ &(d) \quad \frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \mathbf{v} = 0, \quad (e) \quad \frac{\partial p}{\partial t} - a_0^2 \frac{\partial \rho}{\partial t} = 0, \end{aligned}\right\} \tag{3}$$

where $\mathbf{H}, p, \rho$ now denote deviations from $\mathbf{H}_0$, p_0, ρ_0 and

$$a_0^2 = \left(\frac{\partial S/\partial \rho}{\partial S/\partial p}\right)_{p_0, \rho_0} = \left(\frac{dp}{d\rho}\right)_{S=S_0}.$$

From (3e) we see that, except for a function independent of t,

$$\rho = \frac{1}{a_0^2} p. \tag{4}$$

In view of the integrals (2) and (4) the effective order of the system (3) is nine.

3. Plane waves

These were considered in detail first by van de Hulst (1951) and later by Baños (1955) for the simpler case of forced oscillations. Since neither of their treatments is readily adaptable to the present discussion we shall briefly derive the salient features here *ab initio*. In particular, we shall contrast the case of a good conductor (σ large) with that of the more familiar poor conductor (σ small).

Take the x-axis along the direction of propagation and let the x, y-plane contain $\mathbf{H}_0$: then $\mathbf{H}_0 = (H_0 \cos \phi, H_0 \sin \phi, 0)$ where ϕ is the angle at which the waves cross the undisturbed magnetic field. With all variable proportional to $\exp i(\omega t - \kappa x)$ and the same symbols used for the factors, the equations (3) divide into two sets, I and II, involving†

$$\text{I:}\ E_1, E_2, H_3, v_3 \quad \text{and} \quad \text{II:}\ E_3, H_2, v_1, v_2, p\ (\text{and}\, \rho), \tag{5}$$

† Subscripts 1, 2, 3 denote x-, y-, z-components consistently throughout this paper.

respectively. Thus we find from equations $(3b_1)$,† $(3a_3)$, $(3b_2)$, and $(3c_3)$:

$$
\text{(I)}\quad\begin{cases}
E_1 = \dfrac{\mu H_0 \sin\phi}{(1+i\eta\omega/c^2)}\,v_3, \quad E_2 = \dfrac{\mu\omega}{\kappa}\,H_3,\\[2mm]
\left[\omega - i\eta\left(\kappa^2 - \dfrac{\omega^2}{c^2}\right)\right] H_3 + \kappa H_0 \cos\phi v_3 = 0,\\[2mm]
\left(\kappa^2 - \dfrac{\omega^2}{c^2}\right)\left(1+\dfrac{i\eta\omega}{c^2}\right) A_0^2 \cos\phi H_3 + \omega\kappa H_0\left(1+\dfrac{i\eta\omega}{c^2}+\dfrac{A_0^2}{c^2}\sin^2\phi\right) v_3 = 0,
\end{cases}
$$

respectively, where $c^2 = 1/\mu\epsilon$, $A_0^2 = \mu H_0^2/\rho_0$, and $\eta = 1/\mu\sigma$ (magnetic diffusivity). Similarly, from $(3a_2)$, $(3d)$, $(3b_3)$, $(3c_1)$, and $(3c_2)$ we obtain

$$
\text{(II)}\quad\begin{cases}
E_3 = -\dfrac{\mu\omega}{\kappa}\,H_2, \quad p = \dfrac{a_0^2\rho_0\kappa}{\omega}\,v_1,\\[2mm]
\left[\omega - i\eta\left(\kappa^2 - \dfrac{\omega^2}{c^2}\right)\right] H_2 - \kappa H_0 \sin\phi v_1 + \kappa H_0 \cos\phi v_2 = 0,\\[2mm]
\omega\left(\kappa^2 - \dfrac{\omega^2}{c^2}\right) A_0^2 \sin\phi H_2 + \kappa(a_0^2\kappa^2 - \omega^2) H_0 v_1 = 0,\\[2mm]
\left(\kappa^2 - \dfrac{\omega^2}{c^2}\right) A_0^2 \cos\phi H_2 + \omega\kappa H_2 v_2 = 0,
\end{cases}
$$

respectively. The remaining equation $(3a_1)$ gives $H_1 = 0$, in agreement with the requirement (2).

The dispersion relations corresponding to these two systems are

$$
\text{(i)}\quad \frac{\eta^2}{c^4}\omega^4 - \frac{i\eta}{c^2}\left(2+\frac{A_0^2}{c^2}\right)\omega^3 - \left(1+\frac{A_0^2}{c^2}+\frac{\eta^2\kappa^2}{c^2}\right)\omega^2 + i\eta\kappa^2\left(1+\frac{A_0^2}{c^2}\right)\omega + A_0^2\kappa^2\cos^2\phi = 0,
$$

$$
\text{(ii)}\quad \frac{i\eta}{c^2}\omega^5 + \left(1+\frac{A_0^2}{c^2}\right)\omega^4 - i\eta\kappa^2\left(1+\frac{a_0^2}{c^2}\right)\omega^3 - \kappa^2\left(a_0^2 + A_0^2 + \frac{a_0^2A_0^2}{c^2}\cos^2\phi\right)\omega^2 + i\eta\kappa^4a_0^2\omega + \kappa^4a_0^2A_0^2\cos^2\phi = 0.
$$

In the first case there are, in general, four possible values of ω for each given κ, ϕ, and in the second case five. The total of nine corresponds to the eleven original equations less the two t-integrals (2) and (4). We shall see that two of the roots in (i) and one of those in (ii) correspond to pure decays; the remainder (two and four) represent damped sinusoidal waves, which are in general neither transverse nor longitudinal.

From (5) we note that **E** is always perpendicular to both **H** and **v**, these being parallel vectors in case I.

4. The extreme cases

(*a*) *Insulator*. For a perfect insulator $\eta = \infty$, and we find

$$
\text{I:}\quad \omega = \pm c\kappa; \quad E_2 = \pm\sqrt{(\mu/\epsilon)}\,H_3, \quad E_1 = v_3 = 0,
$$

$$
\text{II:}\quad \begin{cases}
\omega = \pm c\kappa; & E_3 = \mp\sqrt{(\mu/\epsilon)}\,H_2, \quad v_1 = v_2 = p = 0,\\
\omega = \pm a_0\kappa; & p = \pm\rho_0 a_0 v_1, \quad E_3 = H_2 = v_2 = 0.
\end{cases}
$$

† $(3b_1)$ is the x-component of equation $(3b)$, etc.

The field $\mathbf{H}_0$ is purely additive. I and the first of II are ordinary (transverse) electromagnetic waves and can be added. The second of II is an ordinary (longitudinal) acoustic wave.

For a good insulator and all but very small wave numbers (i.e. when the non-dimensional quantity $\eta\kappa/c$ is large) these frequencies become (to order $1/\eta^2$)

$$\text{I:}\quad \omega = \pm c\kappa + ic^2/2\eta,$$

$$\text{II:}\quad \begin{cases} \omega = \pm c\kappa + ic^2/2\eta, \\ \omega = \pm a_0\kappa + iA_0^2\sin^2\phi/2\eta. \end{cases}$$

The damping of the electromagnetic waves is independent of the undisturbed state and is the same as for a rigid body. On the other hand, the (much slower) damping of the acoustic wave does depend on the undisturbed state and in particular is zero for $\phi = 0$. In addition, three other roots now occur, given (to order $1/\eta^3$) by†

$$\text{I:}\quad \begin{cases} \omega = i\alpha_1 c^2/\eta; \quad E_1 = [\mu H_0 \sin\phi/(1-\alpha_1)]\, v_3 \doteqdot (\rho_0 c^2/H_0\sin\phi)\, v_3, \\ \omega = i\alpha_2 c^2/\eta; \quad E_1 = [\mu H_0 \sin\phi/(1-\alpha_2)]\, v_3 \doteqdot \mu H_0 \sin\phi v_3, \end{cases}$$

$$\text{II:}\quad \omega = iA_0^2\cos^2\phi/\eta; \quad v_2 \text{ predominates,}$$

where

$$\alpha_1, \alpha_2 = \frac{1}{2}\left(1+\frac{A_0^2}{c^2}\right)\left[1 \pm \sqrt{\left(1-\frac{4c^2A_0^2\cos^2\phi}{(c^2+A_0^2)^2}\right)}\right] \doteqdot 1-\frac{A_0^2}{c^2}\sin^2\phi, \frac{A_0^2}{c^2}\cos^2\phi.$$

The first of I reduces to the familiar charge decay for small H_0 (or ϕ). The other two are then essentially (much slower) velocity decays.

(*b*) *Conductor*. For a perfect conductor $\eta = 0$, and we find

$$\text{I:}\quad \omega \doteqdot \pm(A_0\cos\phi)\,\kappa; \quad E_1/\mu H_0\sin\phi = -E_2/\mu H_0\cos\phi = \mp(A_0/H_0)\,H_3 = v_3 \quad \text{(approx.)},$$

$$\text{II:}\quad \begin{Bmatrix} \omega = \pm V^{(1)}\kappa \\ \omega = \pm V^{(2)}\kappa \end{Bmatrix}; \quad \frac{E_3}{\mu H_0 V} = \frac{\mp H_2}{H_0} = \frac{(a_0^2-V^2)\,v_1}{A_0^2 V\sin\phi} = \frac{Vv_2}{A_0^2\cos\phi} = \pm\frac{(a_0^2-V^2)\,p}{a_0^2A_0^2\rho_0 V\sin\phi} \text{ (approx.)},$$

where $V = V^{(1)}$, $V^{(2)}$ are the positive roots of (approx.)

$$V^4-(a_0^2+A_0^2)\,V^2+a_0^2A_0^2\cos^2\phi = 0.$$

A sketch of $V^{(1)}$ and $V^{(2)}$ as functions of ϕ is given in figure 1. I gives so-called Alfvén waves which progress along the $\mathbf{H}_0$-direction with velocity $\pm A_0$ and with $\mathbf{E}$ perpendicular to $\mathbf{H}_0$ as well as to $\mathbf{H}$. For $\phi = 0$, II represents the same Alfvén waves, now transverse and rotated through 90° about the x-axis, and (longitudinal) acoustic waves with velocity $\pm a_0$. For other values‡ of ϕ such a distinction has meaning only when

$$A_0 \ll a_0: \quad V^{(1)} = a_0, \quad V^{(2)} = A_0\cos\phi. \tag{6}$$

In fact for

$$A_0 \gg a_0: \quad V^{(1)} = A_0, \quad V^{(2)} = a_0\cos\phi, \tag{7}$$

† Of the quantities (5), only those of lowest order are listed here [e.g. in I, $H_3/v_3 = O(1/\eta)$ and hence H_3 is omitted]. Also A_0^2/c^2 will be neglected in comparison with unity whenever appropriate (indicated by 'approx.' or '$\doteqdot$').

‡ For fixed ϕ and $\sqrt{(a_0^2+A_0^2)}$, the root $V^{(1)}$ has a minimum and the root $V^{(1)}$ a maximum when $a_0 = A_0$ (cf. figure 1). At this point the two waves for $\phi = 0$ interchange.

the velocities a_0, A_0 have changed roles, without however a corresponding interchange of the other characteristics of the waves.

For a good conductor and all but very large wave numbers (i.e. when the non-dimensional quantity $\eta\kappa/c$ is small) the damping is determined (approx.) by:

$$\omega^2 - i\eta\kappa^2\omega - \kappa^2 A_0^2 \cos^2\phi = 0,$$

$$\omega^4 - i\eta\kappa^2\omega^3 - (a_0^2 + A_0^2)\,\kappa^2\omega^2 + i\eta\kappa^4 a_0^2\omega + \kappa^4 a_0^2 A_0^2 \cos^2\phi = 0, \tag{8}$$

respectively. From the first of these we find immediately:

$$\text{I:}\quad \omega \doteqdot \begin{cases} \tfrac{1}{2}\kappa[\pm\sqrt{(4A_0^2\cos^2\phi - \eta^2\kappa^2)} + i\eta\kappa] & (\cos\phi > \eta\kappa/2A_0), \\ \tfrac{1}{2}i\kappa[\eta\kappa \pm \sqrt{(\eta^2\kappa^2 - 4A_0^2\cos^2\phi)}] & (\cos\phi < \eta\kappa/2A_0). \end{cases}$$

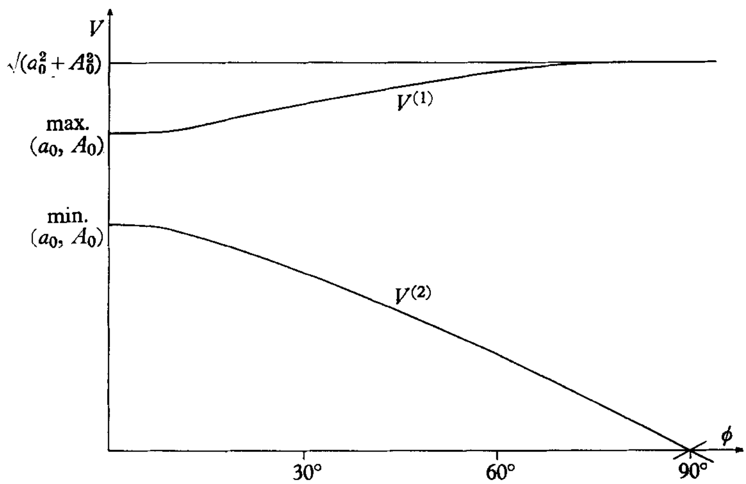

FIGURE 1. The velocities $V^{(1)}$ and $V^{(2)}$ as functions of the angle ϕ between the undisturbed magnetic field and the direction of propagation.

The second yields the damping of the $V^{(1)}$-waves as an η-perturbation. Having found this perturbation, we take out the corresponding factor from the left-hand side of the equation and are left with

$$\omega^2 - i\eta\kappa^2 F(\phi)\,\omega - V^{(2)2} = 0,$$

where

$$F(\phi) = \frac{V^{(1)2} - A_0^2}{2V^{(1)2} - a_0^2 - A_0^2} = \frac{a_0^2 - V^{(2)2}}{a_0^2 + A_0^2 - 2V^{(2)2}},$$

as the equation governing the damping of the $V^{(2)}$-waves. Hence we find

$$\text{II:}\quad \begin{cases} \omega \doteqdot \pm V^{(1)}\kappa + \tfrac{1}{2}i\eta\kappa^2(1-F), \\ \omega \doteqdot \begin{cases} \tfrac{1}{2}\kappa[\pm\sqrt{(4V^{(2)2} - \eta^2\kappa^2F^2)} + i\eta\kappa F] & (V^{(2)}/F > \tfrac{1}{2}\eta\kappa),\dagger \\ \tfrac{1}{2}i\kappa[\eta\kappa F \pm \sqrt{(\eta^2\kappa^2F^2 - 4V^{(2)2})}] & (V^{(2)}/F < \tfrac{1}{2}\eta\kappa).\dagger \end{cases} \end{cases}$$

For I and the second of II there is a range of values near $\phi = \frac{1}{2}\pi$ where the damped waves turn into pure decays. In all cases, when ϕ is small the damping factor changes rapidly near $a_0 = A_0$.‡ There is no damping of the acoustic wave ($\phi = 0$), and for $\phi = \frac{1}{2}\pi$ two of the four decays stop. These last results are in fact exact, see (i) and (ii).

† Using the approximation $V^{(2)} = a_0 A_0 \cos\phi/\sqrt{(a_0^2 + A_0^2)}$ near $\phi = \frac{1}{2}\pi$ these conditions become $\cos\phi \gtrless \eta\kappa a_0/2A_0\sqrt{(c_0^2 + A_0^2)}$.

‡ At $\phi = 0$, F jumps discontinuously from 1 to zero.

As before, there are three more roots now, given (to order η) by

$$\text{I:}\quad \begin{cases} \omega = ic^2/\eta; & E_1/\cos\phi = E_2/\sin\phi, \\ \omega = ic^2(1+A_0^2/c^2)/\eta \doteqdot ic^2/\eta; & E_1/\sin\phi = -E_2/\cos\phi = -(\rho_0 c^2/H_0)\, v_3 \end{cases}$$

$$\text{II:}\quad \omega = ic^2(1+A_0^2/c^2)/\eta \doteqdot ic^2/\eta; \quad (H_0/\rho_0 c^2)\, E_3 = v_1/\sin\phi = -v_2/\cos\phi.$$

Together these correspond to (very fast) decays in $\mathbf{E}$ and $\mathbf{v}\times\mathbf{H}_0$ in which the total momentum $\rho_0\mathbf{v}+(\mathbf{E}\times\mathbf{H}_0)/c^2$ remains constant (zero). We shall return to this point later.

5. Splitting of the general linearized equations

The division of plane waves into two sets has its counterpart for the general flow governed by equations (3). To show this we introduce new Cartesian co-ordinates, where now the y-axis lies along $\mathbf{H}_0$. Also we denote by θ, δ, Δ the divergences of $\mathbf{E}$, $\mathbf{H}$, $\mathbf{v}$ using x- and z-components only:

$$\theta = \frac{\partial E_1}{\partial x}+\frac{\partial E_3}{\partial z}, \quad \delta = \frac{\partial H_1}{\partial x}+\frac{\partial H_3}{\partial z}, \quad \Delta = \frac{\partial v_1}{\partial x}+\frac{\partial v_3}{\partial z}.$$

Then for E_2, θ, $\operatorname{curl}_2\mathbf{H}$, $\operatorname{curl}_2\mathbf{v}$ the following four equations hold:

$$\left.\begin{aligned} \left(\nabla^2-\frac{\partial^2}{\partial y^2}\right)E_2-\frac{\partial\theta}{\partial y}-\mu\frac{\partial}{\partial t}\operatorname{curl}_2\mathbf{H} &= 0,\\ \left(\frac{\eta}{c^2}\frac{\partial}{\partial t}+1\right)E_2-\mu\eta\operatorname{curl}_2\mathbf{H} &= 0,\\ \left(\frac{\eta}{c^2}\frac{\partial}{\partial t}+1\right)\left(\frac{\partial E_2}{\partial y}+\theta\right)+\mu H_0\operatorname{curl}_2\mathbf{v} &= 0,\\ \frac{1}{c^2}\frac{\partial\theta}{\partial t}+\mu\frac{\partial}{\partial y}\operatorname{curl}_2\mathbf{H}-\frac{\rho_0}{H_0}\frac{\partial}{\partial t}\operatorname{curl}_2\mathbf{v} &= 0. \end{aligned}\right\} \tag{9}$$

The first and fourth of these are the y-components of the curl of (3a) and the curl of (3c), respectively; the second is the same component of (3b) and the third is the divergence of that equation.

Similarly, for $\operatorname{curl}_2\mathbf{E}$, H_2, δ, v_2, Δ, p we find the six equations

$$\left.\begin{aligned} \operatorname{curl}_2\mathbf{E}+\mu\frac{\partial H_2}{\partial t} &= 0,\\ \left(\frac{\eta}{c^2}\frac{\partial}{\partial t}+1\right)\operatorname{curl}_2\mathbf{E}+\mu\eta\nabla^2 H_2-\mu H_0\Delta &= 0,\\ \rho_0\frac{\partial v_2}{\partial t}+\frac{\partial p}{\partial y} &= 0,\\ \frac{1}{c^2}\frac{\partial}{\partial t}\operatorname{curl}_2\mathbf{E}+\mu\nabla^2 H_2+\frac{\rho_0}{H_0}\frac{\partial}{\partial t}\left(\frac{\partial v_2}{\partial y}+\Delta\right)+\frac{1}{H_0}\nabla^2 p &= 0,\\ \rho_0 a_0^2\left(\frac{\partial v_2}{\partial y}+\Delta\right)+\frac{\partial p}{\partial t} &= 0,\\ \frac{\partial H_2}{\partial y}+\delta &= 0. \end{aligned}\right\} \tag{10}$$

Here the first and third equations are the y-components of $(3a)$ and $(3c)$ respectively, and the second is the same component of the curl of $(3b)$. The fourth is the divergence of $(3c)$. The fifth is just $(3d)$, after $(3e)$ has been used, and the six this the integrated form of the divergence of $(3a)$ (see equation (2)).

Thus each of $(3a)$—$(3d)$ has given rise to as many equations in (9) and (10) as it has components, and the order (nine) of these in the t-derivative is just two less than that (eleven) of the original system.

By eliminating the other three variables from (9), we find that each of $u = E_2$, θ, $\text{curl}_2\,\mathbf{H}$, $\text{curl}_2\,\mathbf{v}$ satisfies

$$\left[\frac{\partial}{\partial t}\left(\frac{\eta}{c^2}\frac{\partial}{\partial t}+1+\frac{A_0^2}{c^2}\right)\left(\eta\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2+\frac{\partial}{\partial t}\right)-A_0^2\frac{\partial^2}{\partial y^2}\right]u = 0; \tag{11}$$

note that for $\eta = 0$ this is a one-dimensional wave equation. Similarly, by eliminating all but one of $u = \text{curl}_2\,\mathbf{E}$, H_2, δ, v_2, Δ, p from (10) we find

$$\left[\eta\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\left(\frac{\partial^2}{\partial t^2}-a_0^2\nabla^2\right)+\left(1+\frac{A_0^2}{c^2}\right)\frac{\partial^4}{\partial t^4}\right.$$
$$\left.-\left(a_0^2+A_0^2\nabla^2+\frac{a_0^2A_0^2}{c^2}\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2}{\partial t^2}+a_0^2A_0^2\nabla^2\frac{\partial^2}{\partial y^2}\right]u = 0.$$

The two operators reduce to the left-hand sides of equations (i) and (ii), respectively, on setting $\partial/\partial t = i\omega$, $\nabla^2 = -\kappa^2$, $\partial^2/\partial y^2 = -\kappa^2\cos^2\phi$, as they should. For $\eta = 0$, the order of the first in the t-derivative reduces from four to two, while that of second changes from five to four. The combined loss of three corresponds to the fact that $(3b)$ reduces to the algebraic relation

$$\mathbf{E}+\mu\mathbf{v}\times\mathbf{H}_0 = 0 \tag{12}$$

for a perfect conductor, so that $\mathbf{E}$ can no longer be initially prescribed independently of $\mathbf{v}$. For small η the corresponding relaxation of the initial conditions is asymptotically represented by the three modes of decay discussed at the end of the preceding section.†

6. Standing waves in a rectangular box

Suppose the gas fills a rectangular cavity $(x = 0, a;\ y = 0, b;\ z = 0, c)$‡ in a perfectly conducting rigid body into which is 'frozen'§ a uniform magnetic field; let the corresponding external field in the cavity be H_0. Since we must allow the possibility of surface currents for a perfect conductor, the tangential component of $\mathbf{H}$ is not necessarily continuous across the walls of the cavity. Thus, whether or not the gas is considered to be perfect, the complete set of boundary conditions is

$$\left.\begin{aligned} E_2 = E_3 = H_1 = v_1 = 0 \quad &\text{on} \quad x = 0, a,\\ E_3 = E_1 = H_2 = v_2 = 0 \quad &\text{on} \quad y = 0, b,\\ E_1 = E_2 = H_3 = v_3 = 0 \quad &\text{on} \quad z = 0, c,\end{aligned}\right\} \tag{13}$$

† A fuller account has been given by Ludford (1959).

‡ No confusion arises from this second use of c since the old A_0^2/c^2 will be automatically neglected in the remainder of this paper.

§ For a perfect solid conductor Maxwell's equations integrate to give $\mathbf{E} = 0$ and $\mathbf{H}$ independent of t. In the present case this has the effect of isolating the oscillating gas.

expressing the continuity of the tangential component of **E** and of the normal components of $\mu\mathbf{H}$ and $\mathbf{v}$.

By virtue of Maxwell's equation (3*a*) the conditions on **H** are satisfied whenever those on **E** are satisfied. Moreover, for a perfectly conducting gas: $E_2 \equiv 0$, the vanishing of E_3 on the x-faces is implied by the vanishing of v_1, and similarly for E_1 and v_3 on the z-faces [see (12)]. Thus for $\eta \neq 0$ there are only eighteen effective conditions in (13), of which eight are automatically satisfied by virtue of the other ten when $\eta = 0$.

Since the modifying effects of non-zero η have already been sketched in § 4 we restrict the discussion now to $\eta = 0$. Separation of variables then leads to the following solutions of (9):

$$\left.\begin{aligned} H_1 &= \frac{m\pi}{b} H_0 F_z(x,z) \cos m\frac{\pi y}{b} e^{i\omega t}, \\ H_3 &= -\frac{m\pi}{b} H_0 F_x(x,z) \cos m\frac{\pi y}{b} e^{i\omega t}, \\ v_1 &= -\frac{E_3}{\mu H_0} = i\omega F_z(x,z) \sin m\frac{\pi y}{b} e^{i\omega t}, \\ v_3 &= \frac{E_1}{\mu H_0} = -i\omega F_x(x,z) \sin m\frac{\pi y}{b} e^{i\omega t}, \end{aligned}\right\} \tag{14}$$

where in agreement with (i) $\qquad \omega = \dfrac{m\pi}{b} A_0;$ $\qquad$ (15)

in addition $E_2 = H_2 = v_2 = p = 0$. In order to satisfy the boundary conditions on v_1 and v_3, the otherwise arbitrary function $F(x,z)$ must vanish on $x = 0$, a and $z = 0, c$; the sin $m\pi y/b$, with m a non-negative integer, in E_1 and E_3 follows for a similar reason. Note that H and v are 90° out of phase, and that we have ensured $\operatorname{curl}_2 \mathbf{E} = \delta = \Delta = 0$ [equations (10) must also be satisfied].

Similarly, we find product solutions of (10):

$$\left.\begin{aligned} H_1 &= -i\frac{l}{\omega a} H_0 g'(y) \sin l\frac{\pi x}{a} \cos n\frac{\pi z}{c} e^{i\omega t}, \\ H_2 &= \frac{i\pi}{\omega}\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right) H_0 g(y) \cos l\frac{\pi x}{a} \cos n\frac{\pi z}{c} e^{i\omega t}, \\ H_3 &= -i\frac{n}{\omega c} H_0 g'(y) \cos l\frac{\pi x}{a} \sin n\frac{\pi z}{c} e^{i\omega t}, \\ v_1 &= -\frac{E_3}{\mu H_0} = \frac{l}{a} g(y) \sin l\frac{\pi x}{a} \cos n\frac{\pi z}{c} e^{i\omega t}, \\ v_2 &= \pi\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right) f(y) \cos l\frac{\pi x}{a} \cos n\frac{\pi z}{c} e^{i\omega t}, \\ v_3 &= \frac{E_1}{\mu H_0} = \frac{n}{c} g(y) \cos l\frac{\pi x}{a} \sin n\frac{\pi z}{c} e^{i\omega t}, \\ p &= \frac{i\pi}{\omega}\rho_0 a_0^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right) [f'(y)+g(y)] \cos l\frac{\pi x}{a} \cos n\frac{\pi z}{c} e^{i\omega t}, \end{aligned}\right\} \tag{16}$$

where

$$\left.\begin{aligned} a_0^2 f'' + \omega^2 f + a_0^2 g' &= 0, \\ \pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right) a_0^2 f' - A_0^2 g'' + \left[\pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right)(a_0^2+A_0^2) - \omega^2\right] g &= 0, \end{aligned}\right\} \tag{17}$$

and

$$f(0) = f(b) = g(0) = g(b) = 0, \tag{17'}$$

by virtue of the boundary conditions (13) on v_2 and H_2; also $E_2 \equiv 0$. Here l and n are non-negative integers and the $\sin l\pi x/a$, $\sin n\pi z/c$ in v_1, v_3 follow from the boundary conditions, while this time we have to take $\theta = \operatorname{curl}_2 \mathbf{H} = \operatorname{curl}_2 \mathbf{v} = 0$. Note that $\mathbf{H}$ and p are 90° out of phase with $\mathbf{v}$.

On setting f and g proportional to e^{iky} in (17), we find that (ii), with $\eta = 0$, $\kappa^2 = k^2 + \pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right)$, and $\kappa^2\cos^2\phi = k^2$, must be satisfied. For each ω (to be determined), l, n, there are four possible k's, say $\pm k^{(1)}$, $\pm k^{(2)}$, given by the roots of

$$k^4 + \left[\pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right) - \left(\frac{1}{a_0^2}+\frac{1}{A_0^2}\right)\omega^2\right]k^2 + \omega^2\left[\frac{\omega^2}{a_0^2 A_0^2} - \left(\frac{1}{a_0^2}+\frac{1}{A_0^2}\right)\pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right)\right] = 0. \tag{18}$$

In order to satisfy the boundary conditions (17′) with a linear combination of the four pairs of corresponding functions these roots $k^{(1)}$, $k^{(2)}$ must either satisfy

$$\frac{\tan\frac{1}{2}k^{(1)}b}{\tan\frac{1}{2}k^{(2)}b} = \frac{k^{(2)}(a_0^2 k^{(1)2} - \omega^2)}{k^{(1)}(a_0^2 k^{(2)2} - \omega^2)}, \tag{19}$$

in which case (except for an arbitrary constant factor)

$$\left.\begin{aligned} f &= \sin\tfrac{1}{2}k^{(2)}b \sin k^{(1)}(\tfrac{1}{2}b - y) - \sin\tfrac{1}{2}k^{(1)}b \sin k^{(2)}(\tfrac{1}{2}b - y), \\ g &= \frac{a_0^2 k^{(1)2} - \omega^2}{a_0^2 k^{(1)}} \sin\tfrac{1}{2}k^{(2)}b \cos k^{(1)}(\tfrac{1}{2}b - y) - \frac{a_0^2 k^{(2)2} - \omega^2}{a_0^2 k^{(2)}} \sin\tfrac{1}{2}k^{(1)}b \cos k^{(2)}(\tfrac{1}{2}b - y), \end{aligned}\right\} \tag{19'}$$

or else

$$\frac{\tan\frac{1}{2}k^{(1)}b}{\tan\frac{1}{2}k^{(2)}b} = \frac{k^{(1)}(a_0^2 k^{(2)2} - \omega^2)}{k^{(2)}(a_0^2 k^{(1)2} - \omega^2)}, \tag{20}$$

when

$$\left.\begin{aligned} f &= \cos\tfrac{1}{2}k^{(2)}b \cos k^{(1)}(\tfrac{1}{2}b - y) - \cos\tfrac{1}{2}k^{(1)}b \cos k^{(2)}(\tfrac{1}{2}b - y), \\ g &= -\frac{a_0^2 k^{(1)2} - \omega^2}{a_0^2 k^{(1)}} \cos\tfrac{1}{2}k^{(2)}b \sin k^{(1)}(\tfrac{1}{2}b - y) + \frac{a_0^2 k^{(2)2} - \omega^2}{a_0^2 k^{(2)}} \cos\tfrac{1}{2}k^{(1)}b \sin k^{(2)}(\tfrac{1}{2}b - y). \end{aligned}\right\} \tag{20'}$$

Each of the solutions given above involves surface currents, given by the tangential component of $\mathbf{H}$ rotated in the face through 90°, and surface charges, measured by the value of the normal component of $\mathbf{E}$.

7. Distribution of frequencies

The values of ω for the standing waves (14) are given explicitly by (15); they do not depend on a_0, nor on the form of F. On the other hand, the two sets of values of ω for the waves (16) are determined implicitly by (19) and (20), and depend on l, n, a_0, A_0; we shall now determine the nature of these last distributions for the two extreme cases of very weak and very strong magnetic fields $\mathbf{H}_0$.

When $l = n = 0$ equation (18) gives $k^{(1)} = \omega/a_0$, $k^{(2)} = \omega/A_0$ and then (19) and (20) together yield

$$\tan \tfrac{1}{2}k^{(1)}b = 0, \infty; \quad \omega = \frac{m\pi}{b} a_0, \tag{21a}$$

$$\tan \tfrac{1}{2}k^{(2)}b = \infty, 0; \quad \omega = \frac{m\pi}{b} A_0, \tag{21b}$$

where m is a non-negative integer. With these in mind we consider in turn:

(*a*) $A_0 \ll a_0$. For the frequencies with $\omega b/a_0$ of order A_0/a_0 we find from (18)

$$k^{(2)} = \frac{\omega}{A_0}, \quad k^{(1)} = i\pi \sqrt{\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right)} = \frac{iK}{b} \quad \text{(say)},$$

correct to $O(A_0/a_0)$. Thus when l and n are not both zero, equations (19) and (20) reduce to

$$\omega = \frac{A_0}{b}\tilde{\omega}, \quad \tilde{\omega} \begin{matrix}\cot\\ \tan\end{matrix} \tfrac{1}{2}\tilde{\omega} = \pm K \begin{matrix}\coth\\ \tanh\end{matrix} \tfrac{1}{2}K. \tag{22}$$

Graphs of $\tilde{\omega} \cot \frac{1}{2}\tilde{\omega}$ and $\tilde{\omega} \tan \frac{1}{2}\tilde{\omega}$ show that for each l, n there is alternately a root of one of this last pair of equations in every interval $(\pi, 2\pi)$, $(2\pi, 3\pi)$, ..., and that in each interval the roots converge monotonically towards the left end-point as K increases. The large roots must be excluded in forming the corresponding ω's and to the latter we must add (21*b*) which correspond to $l = n = 0$.

On the other hand, when $\omega b/a_0 = O(1)$ we have

$$k^{(1)} = \frac{\omega}{A_0}, \quad k^{(2)} = \sqrt{\left[\frac{\omega^2}{a_0^2} - \pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right)\right]}, \tag{23}$$

correct to $O(1)$. The right-hand side of (19) is now $O(a_0/A_0)$ and may be taken infinite; similarly, the right-hand side of (20) is effectively zero. Thus either

$$\tan \tfrac{1}{2}k^{(1)}b = \infty, 0; \quad \omega = \frac{m\pi}{b} A_0 \quad \text{for all } l, n, \tag{24}$$

or else

$$\tan \tfrac{1}{2}k^{(2)}b = 0, \infty; \quad \frac{\omega^2}{a_0^2} = \pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right). \tag{25}$$

The frequencies (24) apply when m is $O(a_0/A_0)$ (according to our assumption) and give the continuation of (22). The second set (25) is the same as for ordinary acoustic waves in the region.

(*b*) $A_0 \gg a_0$. If $\omega b/a_0$ is $O(1)$ compared with a_0/A_0, we have

$$k^{(1)} = \frac{\omega}{a_0}, \quad k^{(2)} = i\pi \sqrt{\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right)},$$

correct to $O(a_0/A_0)$. Now (19) and (20) become $\tan \frac{1}{2}k^{(1)}b = 0$ and $\tan \frac{1}{2}k^{(1)}b = \infty$, respectively, and so

$$\omega = \frac{m\pi}{b} a_0, \quad \text{for all } l, n. \tag{26}$$

However, when $\omega b/a_0 = O(A_0/a_0)$ the wave numbers are

$$k^{(1)} = \frac{\omega}{a_0}, \quad k^{(2)} = \sqrt{\left[\frac{\omega^2}{A_0^2} - \pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right)\right]},$$

correct to $O(1)$. As before, but this time to a higher degree of accuracy, we may take

$$\tan \tfrac{1}{2}k^{(1)}b = 0, \infty; \quad \omega = \frac{m\pi}{b} a_0 \quad \text{for all } l, n, \tag{27}$$

$$\tan \tfrac{1}{2}k^{(2)}b = \infty, 0; \quad \frac{\omega^2}{A_0^2} = \pi^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right). \tag{28}$$

In the first of these m must, for consistency, be large, and we obtain the continuation of (26).

The most important point to notice is that a_0 and A_0 have now changed roles [compare the remarks concerning (6) and (7)].

8. Limiting forms of the waves

It is of some interest to sketch the forms of the standing waves for the two extreme cases (a) and (b) discussed in the last section.

(a) $A_0 \ll a_0$. The waves (14) give no trouble. From (15) we find $\omega \to 0$ as $A_0 \to 0$ for every m, so that the pairs H_1, H_3 and v_1, v_3 become independent of t (we factor out H_0 to avoid all except E_1 and E_3 becoming zero) and of each other. This is in agreement with the integrals: curl $\mathbf{v}$ = const. (in time) and $\mathbf{H}$ = const. (in time) of equations (3) for $\sigma = \infty$ and $H_0 = 0$. However, we note that in the limit v_1 and v_3 still vanish† on $y = 0, b$ whereas no such boundary conditions are required for $H_0 = 0$ (ordinary acoustic equations govern $\mathbf{v}$). The corresponding vortex sheet which clings to a y-face when $H_0 = 0$ propagates as a transverse wave discontinuity [in the sense of Friedrichs (1957)] with velocity $\pm A_0$ along the y-direction when $H_0 \neq 0$, and suffers successive reflexions at $y = 0, b$. This follows immediately from the fact that $\text{curl}_2\, \mathbf{v}$ satisfies (11), which for $\eta = 0$ is the one-dimensional wave equation.

A similar thing happens in the limit, for the frequencies of order A_0, with the waves (16). The triples H_1, H_2, H_3 and v_1, v_2, v_3 become independent of t and of each other, while E_1, E_3 and p tend to zero, the last ($p \to 0$) following from the fact that $f' + g$ tends to zero like A_0^2 [see the first of equations (17)]. Again the vanishing of v_1 and v_3 on the y-faces occurs and is reconciled with ordinary acoustics in a similar way.

More care is required with the frequencies which are of order unity. According to (23) these involve values of $k^{(1)}$ which tend to infinity like $1/A_0$, so that f and g become indeterminate. The difficulty is due to the order of the limits: $\eta \to 0$, $H_0 \to 0$ and is resolved by interchanging them. Thus for small $\eta \neq 0$ and $A_0 \to 0$, the dispersion relation (8) with $\kappa^2 = \pi^2(l^2/a^2 + n^2/c^2) + k^2$ and $\kappa^2 \cos^2 \phi = k^2$ yields

$$k^{(1)} = \sqrt{\frac{\omega}{2\eta}}(1-i), \quad k^{(2)} = \sqrt{\left[\frac{\omega^2}{a_0^2} - \pi^2\left(\frac{l^2}{a^2} + \frac{n^2}{c^2}\right)\right]}$$

in place of (23). With these, the frequencies in (24) are replaced by

$$\omega = i\frac{m^2\pi^2}{b^2}\eta, \tag{29}$$

† From (12) and the fact that the tangential component of $\mathbf{E}$ must be zero, we see that $\mathbf{v}$ vanishes at any perfectly conducting rigid wall not parallel to $\mathbf{H}_0$.

while (25) are effectively unaltered. The values of ω given by (29) decrease with η and can be expected to tend to zero, so as to give the same results as in the last paragraph. For the remaining values of ω the forms (19′) and (20′) are still correct in the limit $\eta \to 0$, and on dividing through by the exponentially large $\sin k^{(1)}b$, and $\cos k^{(1)}b$, respectively, we find the limiting forms

$$f = -\sin k^{(2)}(\tfrac{1}{2}b-y), \quad -\cos k^{(2)}(\tfrac{1}{2}b-y),$$
$$g = \frac{\pi^2(l^2/a^2+n^2/c^2)}{k^{(2)}}\cos k^{(2)}(\tfrac{1}{2}b-y), \quad -\frac{\pi^2(l^2/a^2+n^2/c^2)}{k^{(2)}}\sin k^{(2)}(\tfrac{1}{2}b-y),$$

except for $y = 0, b$. Then E_1, E_3, H_1, H_2, H_3 vanish in (16) and v_1, v_2, v_3, p are related in the same way as for acoustic waves ($\sin k^{(2)}b$ and $\cos k^{(2)}b$ are respectively zero). Again the limit forms of v_1 and v_3 do not vanish as the y-faces are approached.

(*b*) $A_0 \gg a_0$. The situation here is more straightforward. In (14) the components E_1, E_3 dominate for large H_0, and a high-frequency electric wave results. Similarly, for the frequencies (26), g is $O(a_0/A_0)$ and $f = O(1)$ so that in (16) v_1 and v_3 may be ignored in comparison with the others. For the higher frequencies (27) the component H_2 may be dropped as well. In either case the result is a wave for which the fluid essentially oscillates in the direction of the undisturbed magnetic field. Finally, the frequencies (28) lead to functions f and g which are both $O(1)$. Again the components E_1 and E_3 predominate so that a high frequency electric wave results.

In particular we note that the pressure fluctuations are still carried by the waves with velocity a_0.

9. Orthogonality and the three types of wave

We conclude with a few remarks concerning the solution of the general initial-value problem for the present case.

It follows from the work of Bernstein, Frieman, Kruskal & Kulsrud (1958) that velocity vectors $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ of standing waves with different frequencies are orthogonal,† i.e.

$$\int_0^a\int_0^b\int_0^c [v_1^{(1)}v_1^{(2)} + v_2^{(1)}v_2^{(2)} + v_3^{(1)}v_3^{(2)}]\,dx\,dy\,dz = 0. \tag{30}$$

Thus, assuming that we are dealing with a complete set of functions, we can in principle solve any initial-value problem by expanding the (assumed) given values of $\mathbf{v}$ and $\partial\mathbf{v}/\partial t$ at $t = 0$ in (vector) Fourier series, making use of this set. Note that the initial value of $\partial\mathbf{v}/\partial t$ follows from (3*c*) when H_1, H_2, H_3 are specified (consistently) together with p.

Such an expansion can be broken down into three separate ones, each of which has a different interpretation. From the initial values we first form $\text{curl}_2\,\mathbf{H}$ and $\text{curl}_2\,\mathbf{v}$ and expand them in half-range $(0, b)$ cosine and sine series, respectively, in y. The pairs of coefficients (functions of x, z) then determine separately the pairs of arbitrary functions F in the general standing waves formed from (14): each F is determined by a two-dimensional Poisson equation and zero boundary values on $x = 0, a$ and $z = 0, c$. In this way we find the contribution to the complete solution arising from the frequencies (15).

Next we note that the functions f and g in (19′) are odd and even, respectively.

† This is approximate. In the exact relation the first and last terms in the integrand are multiplied by $(1+A_0^2/c^2)$, where c has its original meaning.

about $y = \frac{1}{2}b$, that the reverse is true for the functions (20′), and that H_2, v_2, Δ, $\partial p/\partial y$ as given by (16) have factors g and f alternately (from the first of equations (17) the factor $f'' + g'$ in $\partial p/\partial y$ is a multiple of f). Moreover, H_2 is in phase (or 180° out of phase) with p and both are 90° out of phase with v_2 and Δ, so that initial values of H_2 and p correspond to one of the coefficients in the general standing wave formed from (16) and initial values of v_2 and Δ correspond to the other.

The procedure is now clear. We form v_2 and Δ from the given initial values and take their odd and even parts (about $y = \frac{1}{2}b$), respectively. The resulting functions are then expanded in half-range $(0, a; 0, c)$ cosine series in x and z, the coefficients being functions of y. Each pair of coefficients $v_2(y; l, n)$ and $\Delta(y; l, n)$ must now be expanded in series whose general terms are of the form $\alpha f(y)$, $\alpha g(y)$ with f and g given by (19′). The value of α is obtained by multiplying the first series by $\pi^2(l^2/a^2 + n^2/c^2) f(y)$ and the second by $g(y)$, adding, and integrating from 0 to b

$$\alpha = \frac{\int_0^b \left[\pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right) v_2(y; l, n) f(y) + \Delta(y; l, n) g(y)\right] dy}{\int_0^b \left[\pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right) f^2(y) + g^2(y)\right] dy}.$$

This follows from the orthogonality relation (30), which for velocity vectors corresponding to pairs of functions $f^{(1)}$, $g^{(1)}$ and $f^{(2)}$, $g^{)2)}$ with the same l, n reduces to†

$$\int_0^b \left[\pi^2\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right) f^{(1)} f^{(2)} + g^{(1)} g^{(2)}\right] dy = 0.$$

The value of α determines one of the coefficients mentioned at the end of the last paragraph and the other follows from a similar analysis of the even part of H_2 and the odd part of $\partial p/\partial y$.

In this way we determine the contribution arising from the frequencies satisfying (19). The remaining part of the solution, corresponding to the frequencies satisfying (20), is obtained from the even parts of v_2, $\partial p/\partial y$ and the odd parts of H_2, Δ using now (20′) for f and g.

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† This relation is easily checked directly from the definitions of f and g.